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Limit cycles for rigid cubic systems [☆]

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Abstract

We consider planar cubic systems with a unique rest point of center-focus type and constant angular velocity. For such systems we obtain an affine classification in three families, and, for two of them, their corresponding phase portraits on the Poincaré sphere. We also prove that for two of these families there is uniqueness of limit cycle. With respect the third family, we give the bifurcation diagram and phase portraits on the Poincaré sphere of a one-parameter sub-family exhibiting at least two limit cycles.

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1. Introduction and main results

The study of the bifurcation diagram of a concrete family of planar vector exhibiting more than one limit cycle is usually a very difficult task. In this paper we consider a family of cubic systems. Our results seem to indicate that it presents at most two limit cycles, but

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we have not been able to prove this assertion. In any case, we have been able to obtain a full description of their phase portraits in some subfamilies having at most one limit cycle, as well as to present the phase portraits of a one-parametric subfamily exhibiting at least two limit cycles.

Let us start giving our motivation to study this family. Planar systems whose angular speed is constant are usually called *rigid* or *uniformly isochronous*. When the origin is nondegenerated they can be written as

$$\begin{cases} \dot{x} = -y + xF(x, y), \\ \dot{y} = x + yF(x, y), \end{cases} \quad (1)$$

where $F(x, y)$ is an arbitrary function in the variables x and y . For these systems the center-focus problem (i.e., the distinction between a focus and a center) is equivalent to the isochronicity problem. Probably, this is one of the reasons for which they haven already studied by several authors, see, for instance, [2–4,15]. One of the simpler systems of the form (1) are

$$\begin{cases} \dot{x} = -y + x(F_0(x, y) + F_m(x, y) + F_n(x, y)), \\ \dot{y} = x + y(F_0(x, y) + F_m(x, y) + F_n(x, y)), \end{cases} \quad (2)$$

where each $F_i(x, y)$ stands for a real homogeneous polynomial of degree i , and $0 \leq m < n$ are fixed natural numbers.

Let us call $\mathcal{H}(m, n) \in \mathbb{N} \cup \{\infty\}$ the maximum number of limit cycles of system (2) in terms of m and n . It is not difficult to prove that

$$\mathcal{H}(0, n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, in [10] a method to decide whether the limit cycle exists for the case n even is given.

By studying the Lyapunov constants of the origin of (2), the lower bounds for $\mathcal{H}(m, n)$, when $m \geq 1$, are obtained in [11]; see Table 1.

The results in Table 1 motivate the main goal of this paper: the study of the case $m = 1$, $n = 2$, where $\mathcal{H}(1, 2) \geq 2$. Notice this is the easiest case for which $\mathcal{H}(m, n)$ is not known. Before describing our results on this case we introduce some notation. Write system (2) when $m = 1$, $n = 2$ as

$$\begin{cases} \dot{x} = -y + x(a + bx + cy + dx^2 + exy + fy^2), \\ \dot{y} = x + y(a + bx + cy + dx^2 + exy + fy^2), \end{cases} \quad (3)$$

where $a, b, c, d, e, f \in \mathbb{R}$, and $d^2 + e^2 + f^2 \neq 0$. To study its phase portrait it is useful to get some normal forms with less parameters. In Proposition 2.1(ii) we will prove that it suffices to consider the following three subcases:

Table 1
Lower bounds for $\mathcal{H}(m, n)$

$m \backslash n$	1	2	3	4
1	–	2	2	3
2	–	–	4	4

$$\begin{aligned}
\text{(I)} \quad & \begin{cases} \dot{x} = -y + x(a + bx + cy + dx^2 + exy + y^2), \\ \dot{y} = x + y(a + bx + cy + dx^2 + exy + y^2), \end{cases} \quad \text{with } d > 0, \\
\text{(II)} \quad & \begin{cases} \dot{x} = -y + x(a + bx + cy + x^2), \\ \dot{y} = x + y(a + bx + cy + x^2), \end{cases} \quad \text{with } c \geq 0, \\
\text{(III)} \quad & \begin{cases} \dot{x} = -y + x(a + bx + cy + dx^2 + exy), \\ \dot{y} = x + y(a + bx + cy + dx^2 + exy), \end{cases} \quad \text{with } d \in \{0, 1\}.
\end{aligned}$$

In fact, the above classification takes into account, after a rescaling, the number of critical points at infinity of the Poincaré compactification of system (3). Families (I)–(III) correspond with 0, 1 and 2 pairs of critical points at infinity, respectively. Furthermore each of these cases corresponds with negative, zero and positive values of $\Delta := e^2 - 4df$, respectively.

Our main result is the following

Theorem 1.1. *The maximum number of limit cycles for families (I) and (II) is one and when it exists it is hyperbolic. Furthermore:*

- (i) *The phase portraits in the Poincaré sphere for families (I) or (II)_{c=0} are given in Fig. 1.*

The bifurcation diagram and the phase portraits in the Poincaré sphere for family (II)_{c>0} are given in Fig. 2. In this figure there appears three bifurcations:

- An Andronov–Hopf bifurcation in the straight line $a = 0$. More concretely, the limit cycle appears for a negative and small enough.*
- An infinite heteroclinic saddle-node connection in the straight line $\xi = 1 + bc + ac^2 = 0$, see Figs. 2.5 and 2.8.*
- An infinite homoclinic saddle-node loop connection in the curve \mathcal{L} which is, for a fixed c , the graph of a function of b . This function gives the only value $a = a^*(b, c)$ for which the corresponding system has an infinite homoclinic saddle-node loop connection, see Fig. 2.2.*

A straightforward consequence of the above theorem is the following corollary.

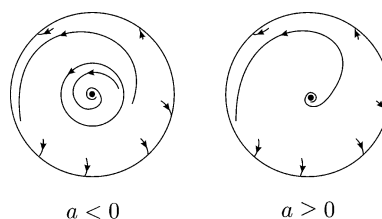


Fig. 1. Phase portraits of families (I) and (II)_{c=0} in accordance with parameter a .

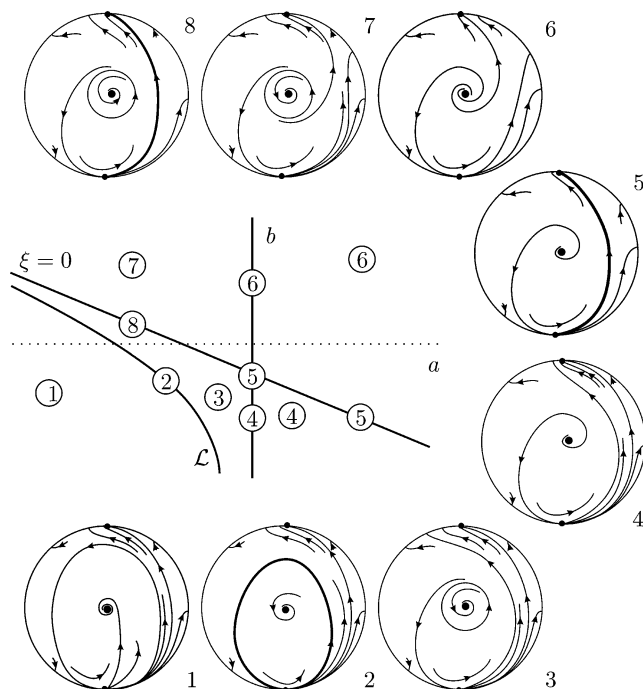


Fig. 2. Bifurcation diagram and phase portraits of family $(II)_{c>0}$ for a fixed value of c . Here $\xi = 1 + bc + ac^2$.

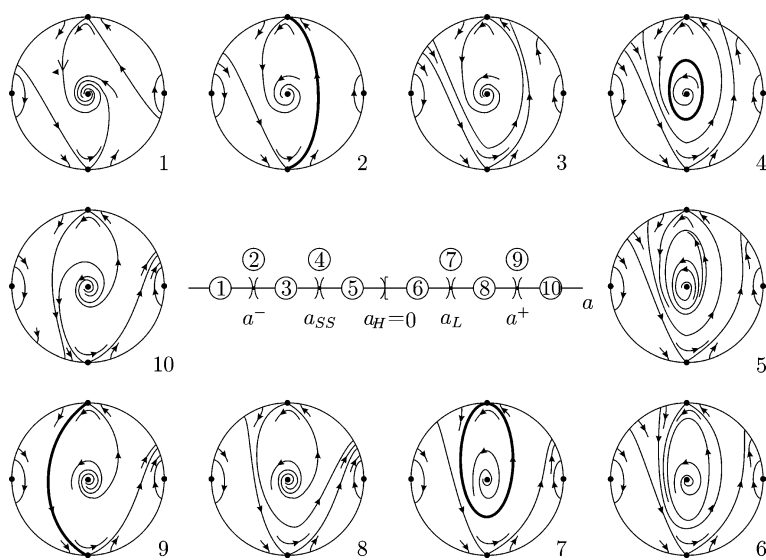


Fig. 3. An evolution of the phase portrait of family (III) with $b = 1$, $c = 3$, $d = 1$ and $e = 5$, in terms of the parameter a exhibiting, at least, two limit cycles. Here $a^- = \frac{-2-\sqrt{29}}{25} \approx -0.29540$, $a_{SS} \approx -0.029101$, $a_L \approx 0.0135$ and $a^+ = \frac{-2+\sqrt{29}}{25} \approx 0.13540$.

Corollary 1.2.

- (i) Families (I) or $(II)_{c=0}$ have limit cycles if and only if $a < 0$.
- (ii) Fixed b and c , there exists a negative value $a^*(b, c)$ such that family $(II)_{c>0}$ has a limit cycle if and only if $a^*(b, c) < a < 0$.

As we will see, a key point for proving the above results will be the fact that system (2), in polar coordinates, writes as an Abel differential equation.

Unfortunately, we have not been able to give an upper bound for the number of limit cycles of system (3). From Table 1 we already know that there are systems of this type with at least two limit cycles. As we have already explained, this result was obtained by using Lyapunov constants. In Section 4 we choose a different approach to get as much limit cycles as it is possible for family (III). Concretely we study the Melnikov functions until order eight of the Hamiltonian system $(x^2 + y^2)/2$ inside the family. We obtain again only two limit cycles, see Section 4. Finally, in Section 5 we illustrate a bifurcation that appears inside family (III) varying only one parameter, see Fig. 3.

2. Preliminary results

As a starting point we will classify system (3) in the three subfamilies (I)–(III).

Let us denote by $\mathcal{P}(X)$ the Poincaré compactification on \mathbb{S}^2 of system (3) (see [12,18], for instance). We note that system (3) has degenerate infinity, i.e., all the equator of \mathbb{S}^2 is filled up of critical points. In order to study the behavior of the orbits when they go to infinity, we reparametrize its Poincaré compactification obtaining, at most, two pairs of critical points at infinity. This number of pairs of infinite critical points will be the criterion to distinguish between the three families in which we will classify system (3). To establish their phase portraits we need, among other things, to know the configuration of the separatrices of their critical points, as well as the exact number of limit cycles that they have. To follow the evolution of the limit cycles and the separatrices when the parameters vary we will deal with semi-complete families of rotated vector fields (see [5] or [16, Section 4.6], for instance).

Next proposition summarizes general properties and technical results on system (3).

Proposition 2.1. *Consider the vector field X associated with system (3) and set $\Delta = e^2 - 4df$. Then:*

- (i) *After a rescaling of the variable t , we obtain zero, one or two pairs of infinite critical points on $\mathcal{P}(X)$, according with $\Delta < 0$, $\Delta = 0$ or $\Delta > 0$, respectively.*
- (ii) *Up to an affine change of coordinates, it is not restrictive to assume that:*
 - (a) *$f = 1$ and $d > 0$ when $\Delta < 0$,*
 - (b) *$e = f = 0$, $d = 1$ and either $c = 0$ or $c > 0$ when $\Delta = 0$,*
 - (c) *$f = 0$ and either $d = 0$ or $d = 1$ when $\Delta > 0$.*
- (iii) *The origin is the unique finite critical point and it is of focus or center type. Furthermore:*

(a) its first three Lyapunov constants are

$$V_1 = e^{2\pi a} - 1, \quad V_3 = \pi(d + f) \quad \text{and} \quad V_5 = \pi((c^2 - b^2)d - bce)/2,$$

(b) the origin is a center if and only if $V_1 = V_3 = V_5 = 0$,

(c) two limit cycles bifurcate from the origin via a degenerated Andronov–Hopf bifurcation.

(iv) System (3) is a semi-complete family of rotated vector fields with respect the parameters a , d or f , indistinctly.

Proof. (i) The expression of system (3) in the local charts \mathcal{U}_1 and \mathcal{U}_2 of the Poincaré compactification of X , given in [12,18], are

$$\begin{cases} \dot{z} = \frac{w}{D(z,w)}(w + z^2w), \\ \dot{w} = \frac{-w}{D(z,w)}(d + ez + bw + fz^2 + czw + aw^2 - zw^2), \end{cases} \quad (4)$$

and

$$\begin{cases} \dot{z} = -\frac{w}{D(z,w)}(w + z^2w), \\ \dot{w} = -\frac{w}{D(z,w)}(f + ez + cw + dz^2 + b zw + aw^2 + zw^2), \end{cases} \quad (5)$$

respectively, where $D(z, w) = \sqrt{z^2 + w^2 + 1}$ and in both cases $w = 0$ represents the infinity of the vector field. Hence, taking a new time variable s such that $ds/dt = -w/D(z, w)$, the infinite critical points, $(z, 0)$, in the local charts \mathcal{U}_1 and \mathcal{U}_2 are given by the roots of $fz^2 + ez + d = 0$ or $dz^2 + ez + f = 0$, respectively. Thus (i) follows.

(ii) Let us prove that if $\Delta \geq 0$, then there is rotation such that $f = 0$ can be assumed. Consider the linear change of coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Easy calculations show that if $\Delta \geq 0$ then, by taking ϕ a real solution of the equation,

$$d \tan^2 \phi + e \tan \phi + f = 0,$$

we get a new system of the form (3) with $f = 0$ as we wanted to prove. The other assertions given in (ii) follow by using a suitable rescaling of the form $\tilde{x} = \alpha x$, $\tilde{y} = \beta y$ and $\tilde{t} = \gamma t$.

(iii) Note that since in polar coordinates $\dot{\theta} \equiv 1$, the origin is the unique finite critical point of system (3). It is also clear that it is of center or focus type. By using the usual methods (see, for instance, [6] or [8]), we get the expressions of V_1 , V_3 and V_5 . To prove that when $V_1 = V_3 = V_5 = 0$ the origin is a center we refer to [3]. An easy example giving rise to two limit cycles is $b = e = 0$, $c = d = 1$, $f = \varepsilon - 1$ with $a > 0$ and $\varepsilon < 0$ and satisfying $|a| \ll |\varepsilon| \ll 1$. Notice that these systems are inside family (III) because $\Delta = 4(1 - \varepsilon) > 0$.

(iv) Writing system (3) as $\dot{x} = P(x, y, \mu)$ and $\dot{y} = Q(x, y, \mu)$, where $\mu \in \{a, d, f\}$, we have

$$\left| \begin{array}{cc} P(x, y, \mu) & Q(x, y, \mu) \\ \partial P(x, y, \mu)/\partial \mu & \partial Q(x, y, \mu)/\partial \mu \end{array} \right| \leq 0,$$

where the equality holds only on noninvariant straight lines or nowhere, depending on the μ chosen. Thus (iv) follows. \square

Remark 2.2.

- (i) In [3] the phase portraits of the centers inside system (3) are given. Observe that from Proposition 2.1(iii), the centers only appear in family (III). Although from our results we can also obtain the results of that paper, we do not repeat that study here.
- (ii) Notice that to assume $f = 0$ in system (3), when $\Delta \geq 0$, means to rotate such system in order to place one infinite critical point at $(0, 0) \in \mathcal{U}_2$.

From Proposition 2.1(i), we can split the study of system (3) into families (I), (II) and (III) according with $\Delta < 0$, $\Delta = 0$ and $\Delta > 0$, respectively.

It will be also useful to have system (3) expressed in (r, θ) -polar coordinates. This is done in the following lemma. Its proof is immediate.

Lemma 2.3. *The periodic solutions of system (3) correspond to positive solutions of the Abel differential equation*

$$\frac{dr}{d\theta} = (d \cos^2 \theta + e \cos \theta \sin \theta + f \sin^2 \theta)r^3 + (b \cos \theta + c \sin \theta)r^2 + ar, \quad (6)$$

that satisfy $r(0) = r(2\pi)$.

Usually the solutions of (6) that satisfy $r(0) = r(2\pi)$ are also called periodic solutions. Furthermore, if they are isolated then are called limit cycles of (6). It is important to notice that in [14], the author proves that there is no upper bound for the number of limit cycles for Abel equations $dr/d\theta = A(\theta)r^3 + B(\theta)r^2 + C(\theta)r$, when A , B and C are arbitrary trigonometrical polynomials.

3. Proof of Theorem 1.1

The goal of this section is to obtain the bifurcation diagrams and the phase portraits on the Poincaré sphere (modulus a change of sign of the independent variable t) of system (3) with $\Delta = e^2 - 4df \leq 0$. By using Proposition 2.1(i) we can take $f = 1$ and $d > 0$ (respectively $e = f = 0$, $d = 1$ and $c \geq 0$) when $\Delta < 0$ (respectively $\Delta = 0$). Notice that we obtain families (I) and (II), respectively.

Next lemma gives the behavior of the orbits of families (I) and (II) near infinity.

Lemma 3.1. *Consider the vector field X associated to system (3) when $\Delta \leq 0$ and $d > 0$.*

- (i) *If $\Delta < 0$, then the equator of its Poincaré compactification, $\mathcal{P}(X)$, is attractor for the flow of $\mathcal{P}(X)$.*
- (ii) *If $\Delta = 0$, after a reparametrization of the time, the singularity $(0, 0)$ in the local chart \mathcal{U}_2 of $\mathcal{P}(X)$ is:*
 - (a) *a cusp point with the two separatrices tangent to $w = 0$, when $c = 0$,*
 - (b) *a saddle-node point, two of whose separatrices tend to $(0, 0)$ in the direction given by the straight line $w = cz$ and the other in the direction $w = 0$, when $c > 0$.*

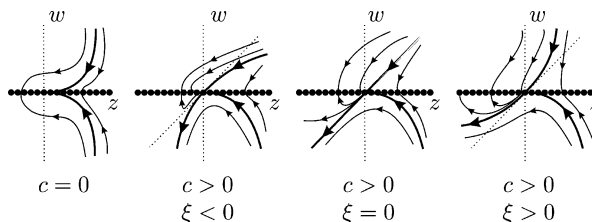


Fig. 4. Phase portraits of family (II) in a neighborhood of $(0, 0) \in \mathcal{U}_2$, in accordance with c and ξ . Recall that $\xi = d + bc + ac^2$.

Furthermore, in the case $\Delta = 0$ and $c = 0$ the infinity is an attractor for the flow. See Fig. 4 for more details.

Proof. If $\Delta < 0$ then, from expressions (4) and (5), we have that $w'|_{\{w=0\}}$ is always positive, where the prime denotes the derivative with respect to s , being $ds/dt = -w/D(z, w)$. Hence, the equator of \mathbb{S}^2 is an attractor for the flow of $\mathcal{P}(X)$.

If $\Delta = 0$ then, from Proposition 2.1(ii), it is enough to consider Eq. (5) with $e = f = 0$ and $d = 1$. In the new time variable s introduced above, this equation becomes

$$\begin{cases} z' = w + z^2w, \\ w' = cw + z^2 + b zw + aw^2 + zw^2. \end{cases}$$

If $c = 0$, we use Theorem 67 of [1] to obtain that the origin, in the local chart \mathcal{U}_2 , is a singularity whose neighborhood is the union of two hyperbolic sectors. After introducing again the time t , we obtain that the equator of \mathbb{S}^2 is attractor for the flow, see Fig. 4.

If $c > 0$, after the change of variables $\tilde{z} = cz - w$, $\tilde{w} = w$, $\tilde{s} = cs$ we can apply Theorem 65 of [1]. We obtain that $(0, 0) \in \mathcal{U}_2$ is a saddle-node, two of whose separatrices tend to $(0, 0)$ in the directions $\pi/2$ and $3\pi/2$ and the other in the direction 0. To obtain Fig. 4, we undo the changes of the variables that we have made. Furthermore, by introducing the parameter $\xi = d + bc + ac^2$, we get that the line $cz - w = 0$ is invariant when $\xi = 0$ and the relative situation of this straight line and two of the separatrices depend on the sign of this parameter, see again Fig. 4. \square

In the next proposition we control the limit cycles of system (3) for families (I) and (II).

Proposition 3.2. Consider system (3) when $\Delta = e^2 - 4df \leq 0$ and $d > 0$. Then it has no centers and has at most one limit cycle. Furthermore, the limit cycle can exist only when $a < 0$ and it is hyperbolic.

Proof. If $a = 0$ then, from Proposition 2.1(iii), we have that $V_3 > 0$ for system (3). Hence, system (3) cannot have a center at the origin. Even more, by taking $a \lesssim 0$ it is possible to generate, as in the proof of Proposition 2.1(iii)(c), one, and only one, small-amplitude limit cycle around it by an Andronov–Hopf bifurcation.

To prove that system (3), under our hypotheses, has at most one limit cycle, let us consider its associated Abel equation given in Lemma 2.3,

$$\frac{dr}{d\theta} = A(\theta)r^3 + B(\theta)r^2 + C(\theta)r,$$

where $A(\theta) = d \cos^2 \theta + e \cos \theta \sin \theta + f \sin^2 \theta$, $B(\theta) = b \cos \theta + c \sin \theta$ and $C(\theta) = a$. Observe that the hypothesis $\Delta \leq 0$ implies that $A(\theta)$ does not change sign. Therefore from [7, Theorem A], we get that Eq. (3) has at most three limit cycles (taking into account their multiplicities) and being one of them $r = 0$. On the other hand, it can be easily checked that if $r(\theta)$ is a periodic solution of Eq. (3) then $-r(\pi + \theta)$ is also a periodic solution, consequently we get that our Abel equation has at most one limit cycle in the half strip $r > 0$, and that when it exists it is hyperbolic. In other words, by using Lemma 2.3, we have proved that system (3) has at most one (hyperbolic) limit cycle.

Let us see that in fact it can only exist only when $a < 0$. From the Andronov–Hopf bifurcation we have seen that the small-amplitude limit cycle is generated when $a \lesssim 0$. By using Proposition 2.1(iv), we know that our system is a complete family of rotated vector fields with respect the parameter a . Hence, the small-amplitude limit cycle born from the origin must expand, covering an annular region, when a is varied in a decreasing way, the outer boundary of it consisting of an unbounded polycycle, i.e., $(0, 0) \in \mathcal{U}_2$ (in the equator of \mathbb{S}^2) belongs to its outer boundary. There is an important property of one-parametric family of rotated vector fields, usually called *nonintersection property*. It reads as follows: limit cycles of distinct vector fields of a semi-complete family do not intersect. Consequently, using the above facts and this property we get that, if $a \geq 0$ system (3) under our hypotheses, does not have limit cycles, because all the region of possible existence of limit cycles is covered by limit cycles existing for the negative values of a . \square

Remark 3.3. In the cases either $\Delta < 0$ or $\Delta = 0$ and $c = 0$ is possible to use an easier argument to conclude that the limit cycle exists only if $a < 0$. From Proposition 3.2 the system has at most one hyperbolic limit and when it exists it is hyperbolic. On the other hand, in these cases, if $a \geq 0$, from the stability of the critical point at the origin, given by Proposition 2.1(iii) and the stability of the equator of \mathbb{S}^2 , given in Lemma 3.1, in case of existence of the limit cycle it would be internally stable and externally unstable. This gives a contradiction with its hyperbolicity. Hence no limit cycles exist for $a \geq 0$.

Next result give us additional information to draw the phase portraits of families (I) and $(II)_{c=0}$. It is a straightforward consequence of Proposition 3.2 and of the above remark.

Lemma 3.4. *When $d > 0$ and either $\Delta < 0$ or $\Delta = c = 0$, the limit cycle of system (3) exists if and only if $a < 0$.*

Collecting all the above results the proof of Theorem 1.1(i) follows.

Remark 3.5. Notice that although the phase portraits of families (I) and $(II)_{c=0}$ are topologically equivalent, there is a difference between them. After a rescaling, in family (I) there are no critical points at infinity, while in family (II) there is a cusp point on the equator.

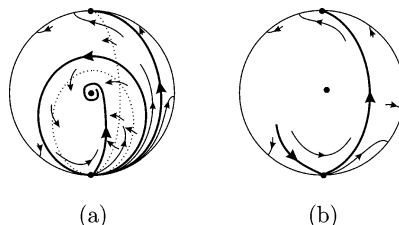


Fig. 5. Phase portrait of family (II) for $c > 0$ and some value of the parameter a exhibiting: (a) that the ω -limit set of the unstable separatrix of $(0, 0) \in \mathcal{U}_2$ is $(0, 0) \in \mathbb{R}^2$, when a is negative enough, (b) $x = 1/c$, as an invariant straight line.

To end the proof of Theorem 1.1 let us consider family (II) when $c > 0$. To control the limit cycle in this case the following result will be useful.

Proposition 3.6. *Let us consider system (3) when $\Delta = 0$, $c > 0$ and $d = 1$. Then, for each pair of real values of b and c there exists a value, $a^* = a^*(b, c) < 0$, such that for $a = a^*$, system (3) has an infinite homoclinic saddle-node loop.*

Proof. Let us fix a pair of arbitrary b and $c > 0$ real parameters in system (3). We note that there exists a value, $a_1 = -(1 + bc)/c^2$ (corresponding to $\xi = 0$ in Fig. 4) for the parameter a such that Eq. (3) has $x = 1/c$, as an invariant straight line. The local chart \mathcal{U}_2 is introduced in the proof of Proposition 2.1. Following [12,18], in this prove we denote by \mathcal{V}_2 the symmetric chart of \mathcal{U}_2 with respect to the origin. The line $x = 1/c$ connects the unstable separatrix of $(0, 0) \in \mathcal{V}_2$ with the stable one of $(0, 0) \in \mathcal{U}_2$. This fact forces that the α -limit set of the stable separatrix of $(0, 0) \in \mathcal{V}_2$ is, from Proposition 3.2, $(0, 0) \in \mathbb{R}^2$ (respectively a limit cycle around $(0, 0) \in \mathbb{R}^2$) if $a_1 \geq 0$ (respectively $a_1 < 0$). See Fig. 5(b), for more details.

The second fact that we will prove is that there exists a value, a_2 , $a_2 < a_1$, for the parameter a for which the ω -limit set of the unstable separatrix of $(0, 0) \in \mathcal{V}_2$ is $(0, 0) \in \mathbb{R}^2$. To prove this it is enough to take into account the following. By one hand, for $\eta_1 = -(2 + bc)/c^2$ and $\eta_2 = (c^4 + 4 + 4bc + b^2c^2)/(4c^3)$ and a negative enough, the parabola $y = -x^2/c + \eta_1x + \eta_2$ is without contact with the flow associated with system (3). More specifically, the subset of \mathbb{R}^2 given by $\{(x, y): y < -x^2/c + \eta_1x + \eta_2\}$ is positively invariant with respect to this flow. Secondly, from Lemma 3.1(ii)(b), if we take a negative enough parameter a in such a way that $\xi = d + bc + ac^2 < 0$, then the unstable separatrix of the saddle-node point is tangent at $(0, 0) \in \mathcal{V}_2$ to the straight line $x = 1/c$ and its ω -limit set must be $(0, 0) \in \mathbb{R}^2$. Let us call a_2 a negative enough parameter a for which the ω -limit set of the unstable separatrix of $(0, 0) \in \mathcal{U}_2$ is $(0, 0) \in \mathbb{R}^2$. See Fig. 5(a), for more details. Hence, since system (3) is a complete family of rotated vector fields with respect the parameter a , there exists a unique value $a^* < 0$, $a_2 < a^* < a_1$, for the parameter a such that system (3) has an infinite homoclinic saddle loop connection. Whence, the proposition follows. \square

Using former results it is possible to draw the phase portraits of family (II) when $c > 0$ as in Fig. 2, according the bifurcation diagram depicted also in Fig. 2. The bifurcation curve

\mathcal{L} gives the infinite homoclinic saddle loop connection and we have obtained it numerically for some values of the parameters. The other curves represented in that figure correspond with the line $\xi = ac^2 + bc + 1 = 0$, where the phase portraits have $x = 1/c$ as an invariant straight line (see Figs. 2.5 and 2.8) and the line $a = 0$, where the phase portraits have an Andronov–Hopf bifurcation at $(0, 0) \in \mathbb{R}^2$ (see Fig. 2.4).

From all the above results part (ii) of Theorem 1.1 follows.

4. Perturbation of the linear center

Consider the next perturbed system

$$\begin{cases} \dot{x} = -y + x \sum_{i \geq 1} \varepsilon^i F_i(x, y), \\ \dot{y} = x + y \sum_{i \geq 1} \varepsilon^i F_i(x, y), \end{cases} \quad (7)$$

with $F_i(x, y) = a_i + b_i x + c_i y + d_i x^2 + e_i x y + f_i y^2$. Notice that it is a subfamily of systems of the form (3). Let (r, θ) be the usual polar coordinates and set $H(x, y) = (x^2 + y^2)/2$. Define $\rho = H(r \cos \theta, r \sin \theta) = r^2/2$. Let $L(\rho, \varepsilon)$ be the return map of (7) associated to the OX^+ -axis parameterized by the energy, ρ . It writes as

$$L(\rho, \varepsilon) = \rho + \varepsilon^k 4\pi L_k(\rho) + O(\varepsilon^{k+1}),$$

being $4\pi L_k(\rho)$ the first nonzero term of its ε -Taylor expansion at 0. The function $L_k(\rho)$ is usually called the k th Melnikov function of system (7). It is well known that the simple positive zeroes of L_k give rise to hyperbolic limit cycles of (7) which tend, when ε goes to zero, to the level curves of H corresponding to these zeroes. For system (7), let us denote by M^k the maximum number of simple positive zeroes of L_k . From Pontryagin's work, see [17], we get

$$\begin{aligned} L_1(\rho) &= \frac{1}{4\pi} \int_{H=\rho} F_1(x, y)(x dy - y dx) \\ &= \frac{1}{4\pi} \int_0^{2\pi} 2\rho F_1(\sqrt{2\rho} \cos \theta, \sqrt{2\rho} \sin \theta) d\theta = (f_1 + d_1)\rho^2 + a_1\rho. \end{aligned}$$

By imposing that $L_1(\rho) \equiv L_2(\rho) \equiv \dots \equiv L_{k-1}(\rho) \equiv 0$, the method developed in [9,13] allows to compute $L_k(\rho)$, for any k . For the first values of k we get

$$\begin{aligned} L_2(\rho) &= (f_2 + d_2)\rho^2 + a_2\rho, \\ L_3(\rho) &= (b_1^2 f_1 - f_1 c_1^2 - c_1 b_1 e_1)\rho^3 + (f_3 + d_3)\rho^2 + a_3\rho, \\ L_4(\rho) &= (-b_1 e_1 c_2 + f_2 b_1^2 - c_1 b_1 e_2 - 2c_1 f_1 c_2 + 2f_1 b_1 b_2 - f_2 c_1^2 - e_1 c_1 b_2)\rho^3 \\ &\quad + (f_4 + d_4)\rho^2 + a_4\rho. \end{aligned}$$

Making all the computations up to order 8 we obtain the main result of this section.

Theorem 4.1.

- (i) For $k = 1, 2, \dots, 8$, the Melnikov functions associated to (7) are polynomials of degree 3 in ρ of the form

$$L_k(r) = m_{2,k}\rho^3 + m_{1,k}\rho^2 + m_{0,k}\rho,$$

where the coefficients $m_{i,k}$ are also polynomials in the coefficients of the polynomials $F_i(x, y)$, $i = 1, 2, \dots, 8$.

- (ii) For system (7) we obtain that $\mathbf{M}^1 = \mathbf{M}^2 = 1$ and $\mathbf{M}^k = 2$ for $k = 3, \dots, 8$.

Notice that the above result shows again that $\mathcal{H}(1, 2) \geq 2$ and provides some more support to the fact that $\mathcal{H}(1, 2)$ is 2.

5. Family (III)

Let us consider system (3) with $\Delta = e^2 - 4df > 0$. From Proposition 2.1(i), its Poincaré compactification has two pairs of critical points at infinity after a rescaling of the variable t . Furthermore, from Proposition 2.1(ii)(c), we can take $f = 0$ (therefore $e \neq 0$) and either $d = 0$ or $d = 1$. Thus we have reduced the study of the case $\Delta > 0$ to the study of family (III).

For this family we have not been able to prove that the maximum number of limit cycles is at most two. However, all our results, presented in Proposition 2.1(iii) and in Section 4, give at most two limit cycles.

For a particular selection of the parameters, we show in Fig. 3, an evolution of the phase portraits of family (III), in terms of the rotatory parameter a . This evolution exhibits, at least, two limit cycles and somehow is representative of the complexity the family. To depict Fig. 3 we need the following results.

The first one is a technical lemma that controls the behavior of the orbits near infinity. Its proof follows from standard techniques and Proposition 2.1.

Lemma 5.1. Consider the vector field X associated with system (3) and the local charts \mathcal{U}_1 and \mathcal{U}_2 , given by (4) and (5), respectively, of $\mathcal{P}(X)$ with coordinates (z, w) , when $\Delta > 0$, $f = 0$, $e \neq 0$ and either $d = 0$ or $d = 1$. Then, after a rescaling of the variable t , $(0, 0) \in \mathcal{U}_2$ and $(-d/e, 0) \in \mathcal{U}_1$ are the singularities at the equator of \mathbb{S}^2 . Even more,

- (i) $(0, 0) \in \mathcal{U}_2$ is a singularity of type: saddle if $e > 0$, unstable (respectively stable) node if $e < 0$, $c^2 + 4e \geq 0$ and $c < 0$ (respectively $c > 0$), unstable (respectively stable) focus if $e < 0$, $c^2 + 4e < 0$ and $c < 0$ (respectively $c > 0$), linear center if $e < 0$ and $c = 0$.
- (ii) $(-d/e, 0) \in \mathcal{U}_1$ is a singularity of type: saddle if $e < 0$, stable (respectively unstable) focus if $e > 0$, $\delta < 0$ and $\sigma < 0$ (respectively $\sigma > 0$), linear center if $e > 0$ and $\delta < 0$ and $\sigma = 0$, stable (respectively unstable) node if $e > 0$, $\delta \geq 0$ and $\sigma < 0$ (respectively $\sigma > 0$), where $\sigma = cd - be$ and $\delta = \sigma^2 - 4e(d^2 + e^2)$.

The second result gives some properties of the phase portrait of system (3), for some selected parameters. Its proof follows from straightforward computations and from the fact that, from Proposition 2.1(iv), system (3) is a semi-complete family of rotated vector fields with respect the parameter a . In particular, the proof of part (ii) is similar to the proof of Proposition 3.6.

Lemma 5.2. *Consider the vector field $X := X_a$ associated to system (3) with $b = 1$, $c = 3$, $d = 1$, $e = 5$ and $f = 0$.*

- (i) *If $a = \frac{-2 \pm \sqrt{29}}{25}$ then X_a has an invariant vertical straight line which equation is $x = \frac{-3 \pm \sqrt{29}}{10}$. See Figs. 3.2 and 3.9.*
- (ii) *There exists a unique value a_L for the parameter a for which X_{a_L} has an infinite homoclinic saddle loop connection. See Fig. 3.7.*
- (iii) *There exists a value a_{SS} for the parameter a for which $X_{a_{SS}}$ has a semi-stable limit cycle. It is externally (respectively internally) stable (respectively unstable). See Fig. 3.4.*

By using again that system (3) is a semi-complete family of rotated vector fields with respect the parameter a and assuming that X_a presents at most two limit cycles, Fig. 3 follows. Some numerical computations show that $a_L \simeq 0.0135$ and $a_{SS} \simeq -0.029101$.

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